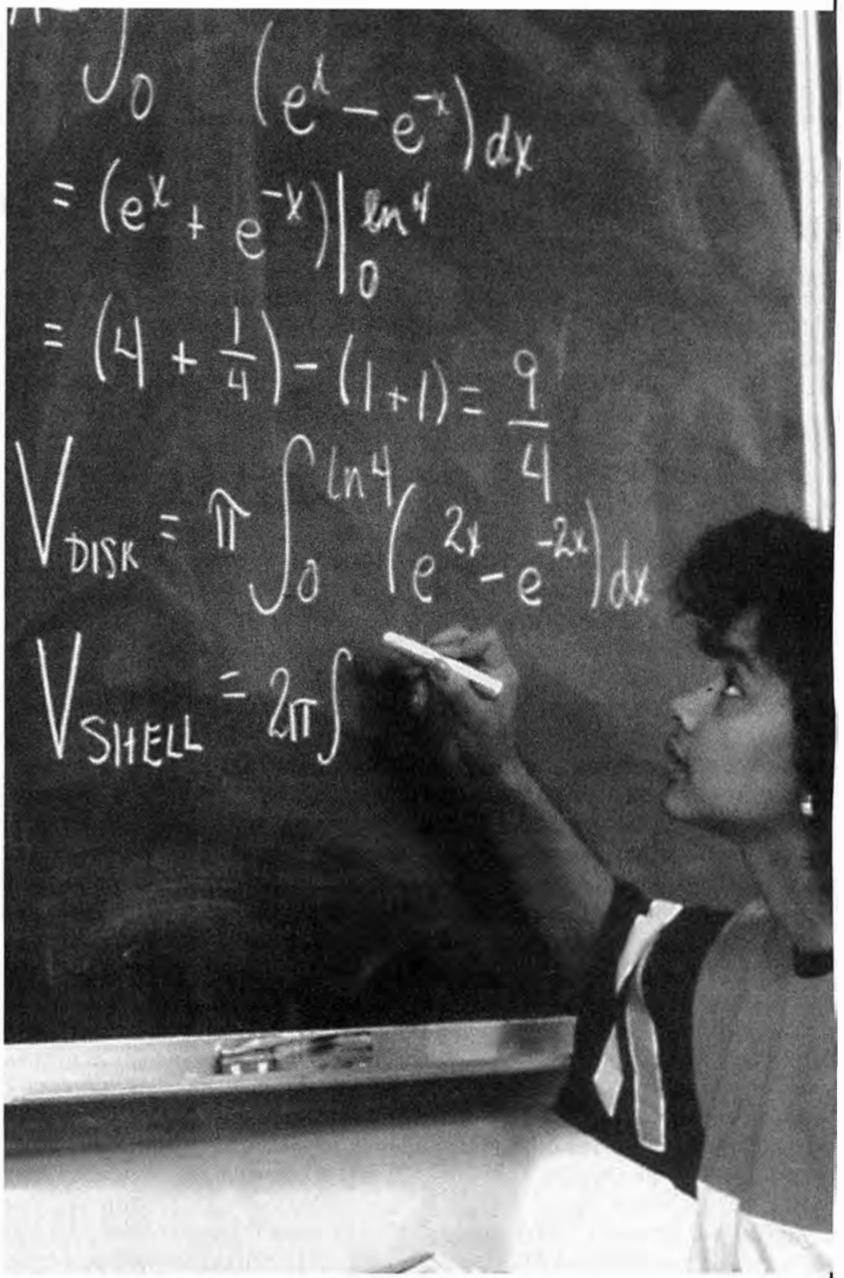
The Real Calculus vs. What You Learned

HOW LEIBNIZ'S ORIGINAL CALCULUS HAS BEEN SUBVERTED

by Ernest Schapiro

A false version of the calculus, based on the Cauchy limit theorem, is now taught in the schools. To revive inventiveness in the physical sciences, students must learn the real creative breakthrough embodied in Leibniz's discovery of the calculus.



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A student in calculus class. But what is she learning?—a set of rules whose discovery has been mystified by the Limit Theorem of Cauchy, or the actual method of invention utilized by Leibniz in discovering the calculus?

Invention of the calculus is one of the greatest discoveries. It permitted the solution to a wide range of mathematical problems by means of a newly invented language or metaphor. It was therefore a great creative breakthrough, and it is therefore entirely predictable that the process by which it was invented has never been properly taught to the millions of people who study the calculus. The basic concept of continuity, which for Gottfried Wilhelm Leibniz (1646–1716) was consistent with the notion of causality, is taught, but in such a way as to stand it on its head. Continuity, rather than being something fundamental, gets defined nowadays as something secondary to "sets of points."

I became interested in the origin of the calculus after hearing a lecture on the subject in Buffalo in 1978. I ordered a book mentioned by the speaker, Carol White, entitled *The History of the Calculus and Its Conceptual Development,* by Carl Boyer.[1] This book, in turn, cited *The Early Mathematical Manuscripts of Leibniz.*[2] When I came to New York City in 1980, I was able to get a copy of the Leibniz work through a company that searches for out-of-print books. After grappling with the book for a few weeks, I could get the main idea of what Leibniz was doing with series. I have been trying since then to figure out why Leibniz, and not other great mathematicians, such as Pascal, Fermat, and Huygens, made the breakthrough. I think the answer to the question requires understanding his philosophical method.

Leibniz, from his teens, was interested in metaphysics and scientific method. His dissertation at age 20, entitled "Dissertation on the Art of Combinations,"[3] concerned the mathematical analysis of complex statements into simpler ones. In the course of his work, he was forced to present his own definitions of commonly used words. In fact, the invention of the calculus was part of a program to enrich the language of Germany. His calculus itself was based upon new poetic metaphors, applied to previously unsolvable problems. He thus enabled everyone to conceptualize something which previously had been extremely difficult.

The Principle of Discovery

Leibniz proposed a project to represent all conceptions of mathematics, law, physical science, and morals by a sort of universal language, which would contain within itself the very principle of discovery. He described this as providing an increase in the powers of reason, comparable to the improvement of vision by the invention of the telescope. He called this the universal characteristic. Unfortunately, he could not enlist the collaboration of any scientists of his time.

However, to break the ground for this project, he developed rigorous definitions, definitions which contained within themselves, wherever possible, the element of causality. He insisted on the principle that the *predicate* is necessarily implied in the *subject.* This were true, whether the truths involved were contingent truths, or necessary, *a priori* truths. A *first truth* is one which predicates something of itself, or denies the opposite of its opposite. For example, *A is A,* or *A is not non-A.* These truths are called *identities.* All other truths are reducible to first truths by the aid of definitions or of concepts.

Leibniz gave as an example the, until then, axiomatic statement: "The whole is greater than the part." Here is how he proceeded:

> "The whole is greater than its part," could be proved by a syllogism, of which the major term was a definition, and the minor term an identity. For, if one of two things is equal to a part of another, the former is called the less, and the latter the greater; and this is to be taken as the definition. Now, if to this definition there be added the following identical and undemonstrable axiom, "Everything possessed of magnitude is equal to itself," i.e. $A = A$, then we have the syllogism:
>
> Whatever is equal to a part of another, is less than that other: (by the definition)
>
> but the part is equal to a part of the whole: (i.e. to itself by identity).
>
> Hence the part is less than the whole. QED[4]

As Leibniz remarked later, this proof was important, because without it, someone would be able to assert an exception to the axiomatic statement. Furthermore, from these considerations came the principle that the *predicate* or *consequent* inheres in the *antecedent.* He restated it as a principle of causality: Nothing happens without a reason. Leibniz wrote:

> In contingent truths however, though the predicate inheres in the subject, we can never demonstrate this, nor can the proposition ever be reduced to an equation or an identity, but the analysis proceeds to infinity, only God being able to see, not the end of the analysis indeed, since there is no end, but the nexus of terms or the inclusion of the predicate in the subject, since He sees everything which is in the series. Indeed this truth arises in part from His intellect and in part from His will, and so expresses His infinite perfection, and the harmony of the entire series of things, each in its own particular way.[5]

As an example of such an infinite series he gave the ratio of the side of the square to the diagonal.

Thus Leibniz's work in mathematics was one aspect of his philosophical program and grand design. He hoped that theological questions could be approached as rigorously as mathematics. In 1679, writing to John Frederick, Duke of Brunswick-Hanover, he said:

> But disputes are more customary than demonstrations in philosophy, morals, and theology, and most readers will have the prejudices about such a project that are usual about works dealing with these matters; for it will be thought that the author has merely transcribed and problematized, and is probably a superficial mind little versed in the mathematical sciences and, consequently, hardly capable of true demonstration. In view of these considerations, I have tried to disabuse everyone by pushing myself ahead a little further than is common in mathematics, where I believe I have made discoveries which have already received the general approval of the greatest men of the day, and which will appear with brilliance whenever I choose. This was the true reason for my long stay in France—to perfect myself in this field, and

From a portrait by Edelinck, David Smith Collection, as in Howard Eves, *An Introduction to the History of Mathematics* (New York: Holt, Rinehart and Winston, 1953)

The young Gottfried Wilhelm Leibniz (right), and Christiaan Huygens (1629-1695). Leibniz's association with the Dutch-born mathematical-physicist in Paris in 1672, set him on the path of discovery of the calculus.

to establish my reputation, for when I went there I was not much of a geometrician, which I needed to be in order to set up my demonstrations in a rigorous way. So I want first to publish my discoveries in analysis, geometry, and mechanics, and I venture to say that these will not be inferior to those which Galileo and Descartes have given us. Men will be able to judge from them whether I know how to discover and to demonstrate. I did not study mathematical sciences for themselves, therefore, but in order some day to use them in establishing my credit and furthering piety.[6]

Series and Differences

In the course of his work with identities, he noted the following case, whose implications had gone unrecognized.

Consider the series of increasing numbers

$A, B, C, D, E,$ and examine the differences

$$A + (B-A) + (C-B) + (D-C) + (E-D) = E$$

$$\quad L \qquad M \qquad N \qquad O$$

$$E-A = L + M + N + O$$

This was identically true of any series of steadily increasing or decreasing numbers. He began to look at some simple series of numbers such as the series of the squares.

0 1 4 9 16 25
1 3 5 7 9

where the second row represents the differences between successive squares. He noticed that the differences of these differences were all 2.

He devised a table of numbers to represent the formations of sums and differences by a kind of shorthand.

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462
1	7	28	84	210	462	924

Looking at this horizontally, any term is the sum of the series to the left just above it. That is, $10 = 1 + 2 + 3 + 4$. Any term is the difference of two just below it and to the left. Furthermore, looking at the diagonals,[7] the terms provide the coefficients for the elevation for $x + 1$ to any power.

For example: $(x+1)^2 = x^2 + 2x + 1$

$(x+1)^3 = x^3 + 3x^2 + 3x + 1$

This has a geometrical interpretation. Thus, to take a square two units on a side and convert it into one with three units on a side, we add on a 1×1 square and two 2×1 rectangles to the original square (Figure 1).

The expression for $(x + 1)^3$ has a geometrical interpretation for cubes.[8] Leibniz's table was a way of representing series of numbers, because each row was constructed by taking the sum of the numbers in the row above, and this principle could be extended as far as one wished. Sums of sums were second sums and differences of differences were second differences. We will see that the notion of a derivative and a second derivative go back to the simple ideas of differences and second differences.

Leibniz looked upon series of numbers as analogous to contingent causal sequences, traceable to an original cause.

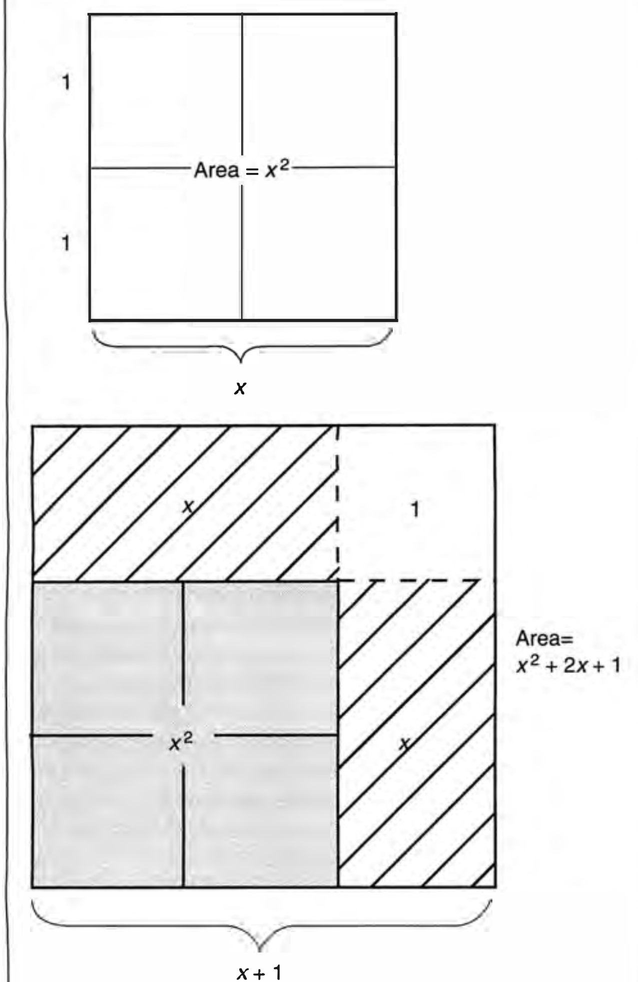


Figure 1

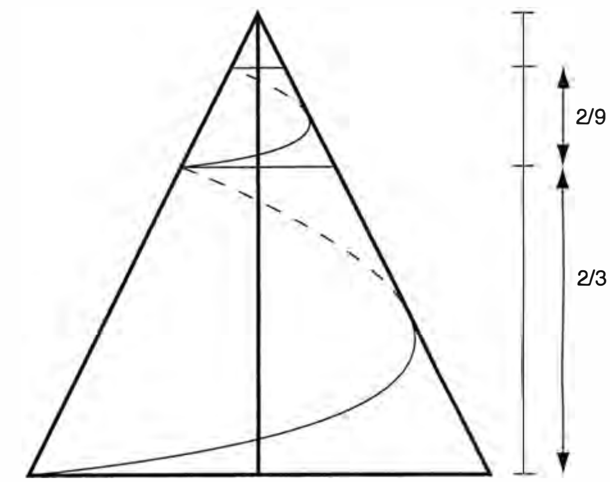


Figure 2

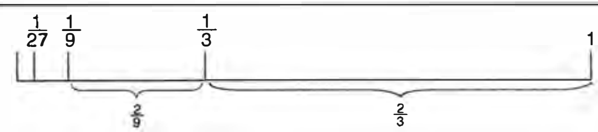


Figure 3

For example, he begins his essay "Art of Combinations," with a proof for the existence of God, based upon all motion in the universe, of necessity, having a first cause. The cause of the sequence may not be apparent on first inspection. However, the generative principle must exist, for nothing happens without a cause. A series of numbers represents a principle of causality. We have already seen how some series have a simple geometric interpretation, such as the series of squares or of cubes.

The series called the geometric series can be considered to represent self-similar growth, as in the formation of a self-similar spiral traversing the surface of a cone from the base to the apex, and always maintaining the same angle to the horizontal (Figure 2). Consider the series

$$1,\ 1/3,\ 1/9,\ 1/27,\ 1/81, \ldots$$

Thus, at the start, the entire height of the cone is yet to be traversed; hence we have 1. After the first turn of a spiral, one-third of the distance remains. After the second turn is completed, only one-ninth remains. Leibniz noticed something interesting about this series, using his new approach. The difference series of a geometric series is, itself, a geometric series. This follows from the self-similar geometry. Leibniz diagrammed the calculation by representing each of the terms as a length; all of these lengths took the same starting point (Figure 3).

Because the first term of the series is 1, and the last term is 0, the sum of all the successive differences between the terms of the series must also equal 1. The successive differences, however, also are a geometric series with the same ratio as the original series!

Leibniz told his colleague Christiaan Huygens, in Paris in 1672, that he had achieved these interesting results with this new principle. Huygens put his young friend to the test, asking him to find the sum of the following continuing series:

$$1 + 1/2 + 1/6 + 1/12 + 1/20 + 1/30 \ldots.$$

Leibniz recognized this series as being the difference series of another series (series A, below); and, this allowed for its sum to be readily determined. Here is how he worked it:

$$\text{series } A = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7} \ldots$$

$$\text{series } B = \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12},\ \frac{1}{20},\ \frac{1}{30},\ \frac{1}{42}, \ldots$$

Leibniz was aware that series A was not convergent, that is, its sum was infinity, not a particular number. Therefore, he cut series A off after n terms. This means that there are $n - 1$ terms in the B series of differences. Leibniz discovered that the sum of these $n - 1$ differences is equal to $1 - (1/n)$. (The reason for this is found in the rule Leibniz discovered in his study of identities, mentioned earlier, that the sum of the differences is equal to the difference between the first and last terms of the original series.) So, for example, consider the sum of series *B*, up through the third term, 1/12. This is the $n - 1$ term, so *n* would equal 4. Then the sum of series *B* up through this term should be $1 - 1/4 = 3/4$. Adding the three terms shows that it is, and the same holds for any term. Now, if we take the expression $1 - (1/n)$, describing the sum of series *B*, and consider it as *n* gets larger and larger, we see that $1/n$ gets very small. Thus the expression for the sum of series *B* approaches 1. This was the answer to Huygens's test.

$$1$$

$$\tfrac{1}{2} \quad \tfrac{1}{2}$$

$$\tfrac{1}{3} \quad \tfrac{1}{6} \quad \tfrac{1}{3}$$

$$\tfrac{1}{4} \quad \tfrac{1}{12} \quad \tfrac{1}{12} \quad \tfrac{1}{4}$$

$$\tfrac{1}{5} \quad \tfrac{1}{20} \quad \tfrac{1}{30} \quad \tfrac{1}{20} \quad \tfrac{1}{5}$$

$$\tfrac{1}{6} \quad \tfrac{1}{30} \quad \tfrac{1}{60} \quad \tfrac{1}{60} \quad \tfrac{1}{30} \quad \tfrac{1}{6}$$

$$\tfrac{1}{7} \quad \tfrac{1}{42} \quad \tfrac{1}{105} \quad \tfrac{1}{140} \quad \tfrac{1}{105} \quad \tfrac{1}{42} \quad \tfrac{1}{7}$$

Figure 4
HARMONIC TRIANGLE

Leibniz saw that series of fractions, just like the series of integers, could also be derived, *ad infinitum,* from one another. He constructed another table, which he called the harmonic triangle (Figure 4). This was based upon the same rule, namely, that successive rows were composed of the difference terms of the previous row. (Thus, 1/2 is the difference between 1 and 1/2; 1/6 is the difference between 1/2 and 1/3; 1/12 is the difference between 1/3 and 1/4, and so forth.)

Leibniz began to think about how this approach, which was valid for integers and fractions, might also be valid for series of infinitesimally small numbers. We will soon see how that was applied.

Huygens was delighted by Leibniz's discovery. The particular series he had asked Leibniz to solve had already been worked out by Hudde. But the approach Leibniz took was original. Huygens asked Leibniz to study geometry, especially the determination of the areas of surfaces of revolution. Leibniz read the writings of Blaise Pascal. In particular he was fascinated by Pascal's solution to the surface of a sphere, conceiving of the sphere as produced by the rotation of a circle about an axis. Figure 5 is the diagram which Pascal constructed to represent the solution to the surface generated by rotating a quadrant of a circle about an axis. Pascal was able to transform the surface of the hemisphere generated by this rotation into a rectangle. This section represents a stage in Leibniz's efforts to develop the calculus; it does not embody the basic conception which he later achieved. If it is too difficult for the reader, don't worry too much, just go on to the next section.

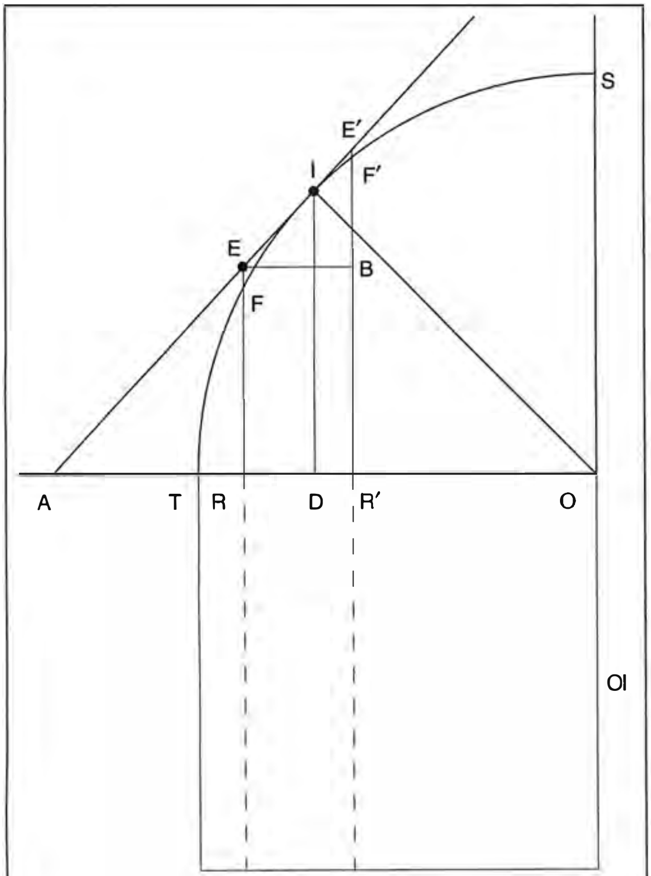


Figure 5

In this figure, *OI* is a radius. The vertical strip with base *RR′* is actually of *infinitesimal* width. *I* is some point located vertically above the width *RR′*. *EB* is equal to *RR′*. *EE′* is the tangent to the circle at the point I. By the tangent, we mean a line touching the circle at one and only one point. Then we can show that the little infinitesimal triangle, *EE′B*, and the triangle *OID* are similar. (The line *ID* divides right triangle *AIO* into triangles *IAD* and *OID,* which are similar to each other, as well as to triangle *AIO*. That is, they have the same three angles, and therefore their sides are proportional, or, in Leibniz's description, they are indistinguishable apart from their size. *EE′B* and *IAD* are similar because their sides are parallel. Because *IAD* and *OID* are similar, so are *EE′B* and *OID*.)

Based on this similarity of *EE′B* and *IOD*, Pascal concluded that $EE' \times DI = RR' \times OI$ (the radius), and that this relationship must hold for each vertical infinitesimal strip! To find the surface for the entire hemisphere, we need the surface generated by rotating the quadrant about the *OR′R* axis. Each vertical strip or *sinus*, such as *RR′*FF′, when rotated about the base, will generate a circular band upon the hemisphere of arc length *FF′*, that is, an arc length very close to the length of the tangent *EE′*. Pascal then said, that if we were to take the entire quadrant as divided up into these infinitesimally thin vertical strips, then

$$\Sigma\, EE' \times DI = OI^2,$$

where Σ denotes a process of summation. We get OI^2 on the right side, because *OI* is being multiplied in succession by each of the lines *RR'*, from *O* out to *T*, and their sum is also *OI*.

But what is the product $EE' \times DI$? It is the area of a cylinder of approximate radius *DI* and height *EE'*, provided we also multiply by 2π. We say *approximate radius*, because *DI* lies between the two diameters of the little cylinder, *RE'* and *R'E'*. The total surface of the hemisphere is obtained by summing up all of these little cylinders. Since the two radii are not exactly equal, that is, *RE* and *R'E'*, these are not perfect cylinders. This was justified, because as the vertical strip gets thinner and thinner, the tangent line *EE'* comes closer and closer to being equal to the arc of the circle *FF'*. Therefore, the area of the infinitesimal cylinder becomes equal to the area of the infinitesimal circular band on the surface of the sphere generated by rotating the quadrant around the axis *AO*. It gives the result: 2π times the radius squared. Notice that what we were also doing was to construct a rectangle of base equal to the sum of all the *RR'*s and of constant height *OI*. Because we were summing the *RR'*s all the way out to the end, the rectangle is, in this case, a square. This is illustrated by the strips placed vertically below the line *OA*. Thus, we have, in fact, been converting the surface of the sphere into a plane area, in this case a square.

Now Leibniz was suddenly struck by the observation that this method, which Pascal had limited to the sphere, could actually be used for any surface of revolution. In this case, the plane area would be constructed as before by taking the normal (perpendicular) to the curve at a given point on the curve. Whereas, in the case of the sphere, the normal was always the radius of the circle, in the case of some other surface of revolution, say a paraboloid, the normal would be of varying length. However, one could still derive the characteristic triangle for the curve at each point and erect below that point, as before, a perpendicular, not to the curve but to the axis of rotation below the curve, and of length equal to the original normal to the curve. One then had the difficult task of summing all the rectangular strips.

Generating a Curve

Leibniz spent some time working out solutions based upon this new approach, which, it turns out, was also being utilized by Barrow, Newton's teacher. Although this method used the tangent to the curve, it was not until 1676 that Leibniz began to use the method of differences to derive tangents. In that year he made a crucial breakthrough, when he realized that the determination of the tangent to the curve could be obtained very easily by use of the principles he had already been applying with series of integers and series of fractions. He also realized that, because determining the tangent to a curve was equivalent, as we shall see, to finding the successive differences of the curve, then, because finding areas of surfaces involved a process of summation of a series, it amounted to an inverse tangent problem. That is to say: Given a function or curve, determine that second function for which the first function or curve was the tangent. If this sounds very complicated, just take another look at the arithmetic and harmonic triangles. Recognize again, that summation, and the taking of successive differences, are the inverse of one another. The principle is, in fact, childishly simple—but only a great creative genius was able to see its application, as we are about to demonstrate.

Leibniz saw that the characteristic triangle, *BEE'* used in Pascal's calculation of the sphere, reflected not just the property of the curve at that point, but, of necessity, the process of generation of the entire curve, of which the point was only one moment. Therefore, he looked at the process governing the generation of the curve from the same standpoint from which he had looked at the formation of all other series.

Consider the parabola with equation $y = kx^2$ (Figure 6). This equation for the parabola, and the equations of other conic sections, were already known at this time, and Leibniz was reading about them in the works of Descartes. The line from x_0, passing through $(x_1 y_1)$, and reaching the vertical line on the right is the tangent to the parabola. The line from $(x_1 y_1)$ to $(x_2 y_2)$ is a chord of the parabola.

The tangent in this parabola can be represented by its slope. For instance, the slope of the first line can be represented by

$$\frac{y_1 - y_0}{x_1 - x_0}.$$

This line is the tangent at (x_1, y_1).

The slope of the line joining the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

We thus can think of the tangent as being the first of the series of such lines connecting point (x_1, y_1) with a series of

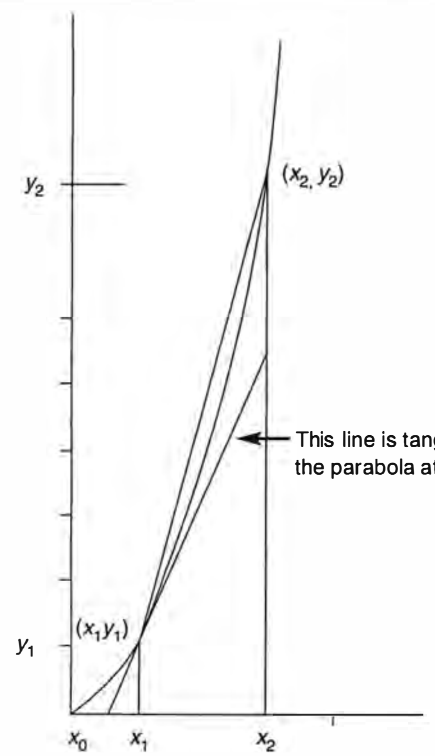


Figure 6
THE PARABOLA $y = kx^2$

Two leading opponents of the school of Descartes, the mathematician-philosophers Pierre de Fermat (left, 1601-1665) and Blaise Pascal (1623-1662), laid the foundation, through their work in number series and geometry, for Leibniz's discovery.

points further up along the parabola. Because it is the first such line, it connects the point (x_1, y_1) with itself. Leibniz saw that the successive values of the slopes of these lines formed a series, and that, if he could determine their rule of formation, then he could deduce the value of the series at the starting point. He looked at the successive differences of these values of the slope by the following simple calculation. By the known equation of the parabola,

$$y_1 = kx_1^2$$

Then, if $x_2 - x_1 = dx$,

$$y_2 = k(x_1 + dx)^2$$

(also by the known equation of the parabola).

Then, if $dy = k(x_1 + dx)^2 - kx_1^2$,

$$dy = y_2 - y_1 = k(x_1^2 + 2x_1 dx + dx^2) - kx_1^2$$

Note that dy and dx denote hypothetical changes in y and x; we are conducting a thought experiment for which Leibniz was prepared to provide full justification.

Then what was the value of the slope of the curve at the point $x_1 y_1$? Taking the ratio, we get

$$\frac{dy}{dx} = k(2x_1 + dx)$$

Here Leibniz introduced his principle of continuity. He conducted a thought experiment. His principle stated:

> In any supposed transition ending in any terminus, it is permissible to institute a general reasoning in which the final terminus may also be included.[10]

The use of these new kinds of numbers, he compared with the successful use of the imaginary numbers:

> It will be sufficient if, when we speak of infinitely great (or, more strictly, unlimited), or of infinitely small quantities (i.e. the very least of those within our knowledge), it is understood that we mean quantities that are indefinitely great or indefinitely small; i.e., as great as you please, or as small as you please, so that the error that anyone may assign may be less than a certain assigned quantity. Also, since in general it will appear that, when any small error is assigned, it can be shown that it should be less, it follows that the error is absolutely nothing; an almost exactly similar kind of argument is used in different places by Euclid, Theodosius, and others; and this seemed to them to be a wonderful thing, although it could not be denied that it was perfectly true that, from the very thing that was assumed an error, it could be inferred that the error was nonexistent. Thus, by infinitely great and infinitely small, we understand something indefinitely great, or something indefinitely small, so that each conducts itself as a sort of class, and not merely as the last thing of a class. If any one wishes to understand these as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even

> though he think that such things are utterly impossible; it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit. For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner by the method already stated.[11]

In other words, we can include the case of $dx = 0$. As we will see, this approach of Leibniz evoked howls of protest: "How can you divide by zero?" Here is another formulation Leibniz gave of this principle:

> If in a given series one value approaches another value continuously, and at length disappears into it, the results dependent on these values in the unknown series must also necessarily approach each other continuously, and, at length, end in each other. So in geometry, for example, the case of an ellipse continuously approaches that of a parabola, as one focus remains fixed and the other is moved farther and farther away, until the ellipse goes over into a parabola when the focus is removed infinitely. Therefore, all the rules for the ellipse must of necessity be verified in the parabola (understood as an ellipse whose second focus is at an infinite distance.) Hence, rays striking a parabola in parallel lines can be conceived as coming from the other focus, or tending towards it.[12]

(Remember that when a light source is placed at one focus of an ellipse, the light is reflected back to the other focus. When a light source is placed at the focus of a parabolic mirror, it is reflected out in parallel rays; when parallel rays strike a parabolic mirror, they are reflected back through the focus of the parabola.)

Leibniz's solution is based upon the method of hypothesis, of a thought experiment in which a universal principle is invoked. As we shall see, it is this method of hypothesis to which his adversaries objected. Through the method of hypothesis, he had brought into existence a new kind of number, denoted by a metaphor, *dy/dx,* which has permanently enriched our language. Even his most bitter adversaries have been forced to adopt the metaphor in doing their calculations, although they have tried to mystify the way it was invented.

Once the rules of obtaining the tangent for a particular type of function or curve are worked out, the rest is child's play. For example, the derivative or tangent to the exponential x^n is $nx^{(n-1)}$. Leibniz also deduced the derivatives for first derivatives, namely, the second derivatives. In this, he was entirely unique; tangents for certain curves had already been discovered, but no one had worked out, or even conceived of, second derivatives. The science of wave motion, and much of mathematical physics, requires the second derivative.

The Principle of Continuity

Textbooks of calculus describe this tangent-determining process as equivalent to finding the derivative, or *dy/dx,* at the point. However, rather than utilizing the principle of continuity, they make continuity itself a secondary idea, one that is deduced from sets of points. The tangent is referred to as a limit obtained as one approaches, but never quite reaches, the point. This is in contradiction to Leibniz, who stated clearly that the end-point or terminus of the process must be included in the process. The Leibnizian approach enables us to see the growth process in the curve. By the principle of continuity, we can and must relate changes in the discrete to changes in the continuous manifold where causality is located. For example, the difference series for cubes shows us how cubes grow by adding on squares, lines, and points. The calculus, for the first time, helps us to hypothesize what must be going on in the continuous manifold between the moments when the new singularities pop out, that is, when the new layers are added onto the faces of the cube.

All growth processes generate a series of numbers. These series, in turn, are a means of describing the original process. As we remarked earlier, Leibniz saw that the inverse tangent calculation could be used to determine surfaces and areas. This amounted to simply determining what the series of numbers must be for which the first series constituted the first differences. This very easy approach gave Leibniz solutions to very difficult, or hitherto unsolved, problems. For example, Archimedes worked out a very tedious solution to the area under a parabola; his method is called the method of exhaustion, and well it might be, because it is so tedious! As we shall see, Leibniz's method makes use of his new language to solve the problem almost instantly.

Consider the series of strips of infinitesimal width, dx (Figure 7). Then the area of the rectangular strip of height kx^2 and the width dx is kx^2dx. Now, since these strips form an increasing series, it must be that there exists a second series for which they are in turn the differences. How did Leibniz figure out what that series is? Very simple: just take the inverse of the difference-forming process. The series of cubes has its differences in the form

$$(x + dx)^3 - x^3 = 3x^2dx + 3xdx^2 + dx^3.$$

When dx is made infinitesimally small, then, because dx^2 is incomparably bigger than dx^3, and incomparably smaller than dx, this reduces to $3x^2dx$.

Therefore, for the parabola, $y = kx^2$, the function

$$\tfrac{1}{3}kx^3$$

gives the series kx^2dx as its difference series. By making the rectangles infinitesimally narrow, their sum gives an increasingly close approximation to the area under the curve. Remember Leibniz's original discovery, that the sum of any series of differences equals the difference between the first and last terms of the second series, which gives rise to those difference. Therefore, the sum of the differences kx^2dx is equal to the value of $\frac{1}{3}kx^3$ at the right-hand endpoint, minus its value at the left-hand endpoint. Today this is called the *definite integral.*

The Limits of Courant

Having been through this demonstration of Leibniz's method, you may be thinking that surely it excites admiration among today's mathematicians, and is taught and used as a model for students. Wrong! All you need to do is to

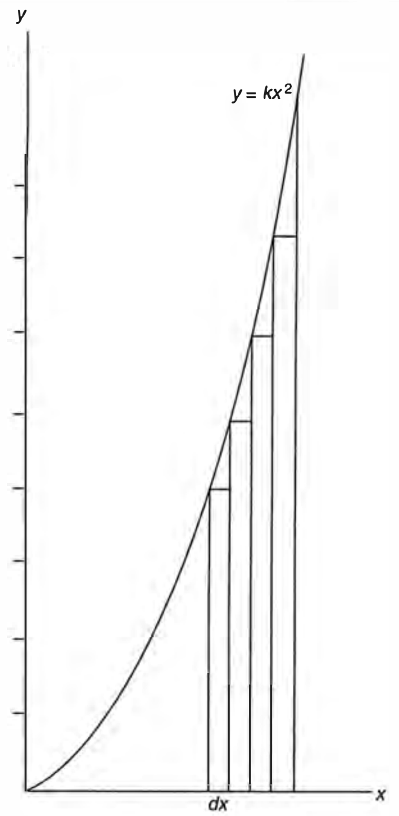


Figure 7
THE AREA UNDER A PARABOLA

examine the vicious slanders and distortions in the following commentaries on Leibniz, which, like iron filings in a magnetic field, point along the controlling lines of force. Let us look at the famous textbook *What Is Mathematics?* by Richard Courant, who was director of the prestigious Institute for Mathematical Sciences at New York University. He writes of Leibniz:

> His achievement is in no way diminished by the fact that it was linked with hazy and untenable ideas which are apt to perpetuate a lack of precise understanding in minds that prefer mysticism to clarity.

And further:

> In the mathematical analysis of the seventeenth and most of the eighteenth centuries, the Greek ideal of clear and rigorous reasoning seemed to have been discarded. "Intuition" and "instinct" replaced reason in many important instances.[13]

Leibniz is misrepresented, and his concept of continuity is omitted, in a later section of Courant's book entitled, "Leibniz' Notation and the 'Infinitely Small.'" Courant there reduces Leibniz's powerful metaphor, *dy/dx,* to a "symbolic notation," so as to leave out the underlying idea. Courant even implies that Leibniz really meant the same thing as he:

> Leibniz's attempt to "explain" the derivative started in a perfectly correct way with the difference quotient of a function $y = f(x)$,
>
> $$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x)}{x_1 - x}$$
>
> For the limit, the derivative, which we called $f'(x)$ (following the usage introduced later by Lagrange), Leibniz wrote *dy/dx*, replacing the difference symbol Δ by the "differential symbol" *d*. . . .

After insisting that we can avoid the problem of dividing by $dx = 0$, if and only if we resort to the "limiting process," Courant attacks Leibniz:

> Mystery and confusion only enter if we follow Leibniz and many of his successors by saying something like this: "Δ*x* does not approach zero. Instead, the 'last value' of Δ*x* is not zero, but an 'infinitely small quantity,' a 'differential' called *dx;* and similarly, Δ*y* has a 'last' infinitely small value *dy*. . . ." Such infinitely small quantities were considered a new kind of number, not zero but smaller than any positive number of the real number system. Only those with a real mathematical sense could grasp this concept, and the calculus was thought to be genuinely difficult, because not everybody has, or can develop, this sense.[14]

Courant's criticism is basically that which Leibniz's work encountered from the time that it first appeared. However, the replacement of the *principle of continuity* by the idea of *limits* was codified in the 19th century by Augustin Cauchy, and this is the view espoused by Courant. Cauchy was deployed against Leibniz and his entire tradition of Continental Science. Cauchy's approach is the one taught in today's mathematics classes all over the world. It is responsible for mystifying the calculus and making it so difficult to learn, especially the differential calculus.

Carl Boyer, author of *The History of the Calculus and Its Conceptual Development,* was a student of Courant. He is outraged at the idea that Leibniz's description represents physical reality. He denies that instantaneous velocity at a point, represented by the tangent at the point, actually exists. Rather, he says, the instantaneous velocity is the limit which the average velocity, referred to above by Courant, approaches as the intervals get small enough:

> Inasmuch as the laws of science are formulated by induction on the basis of the evidence of the senses, on the face of it there can be no such thing in science as an instantaneous velocity, that is, one in which the distance and time intervals are zero. The senses are unable to perceive, and science is consequently unable to measure, any but actual changes in position and time. The power of every sense organ is limited by a minimum of possible perception. We cannot, therefore, speak of motion or velocity, in the sense of a scientific observation, when either the distance or the corresponding time interval becomes so small that the minimum of sensation involved

in its measurement is not excited—much less when the interval is assumed to be zero. . . .

This difficulty has been resolved by the introduction of the derivative, a concept based on the idea of the limit. In considering the successive values of the difference quotient $\frac{\Delta s}{\Delta t}$ [distance over time —ed.], mathematics may continue to indefinitely make the intervals as small as it pleases. In this way, an infinite sequence of values r_1, r_2, r_3, . . . r_n, . . . (the successive values of the ratio $\frac{\Delta s}{\Delta t}$) is obtained. This sequence may be such that the smaller the intervals, the nearer the ratio r_n will approach to some fixed value L, and such that by taking the value of n to be sufficiently large, the difference $|L - r_n|$ can be made arbitrarily small. If this be the case, this value L is said to be the limit of the infinite sequence, or the derivative $f'(t)$ of the distance function $f(t)$, or the instantaneous velocity of the body. It is to be borne in mind, however, that this is not a velocity in the ordinary sense and has no counterpart in the world of nature, in which there be no motion without a change in position.[15]

On Leibniz's principle of continuity, Boyer says:

. . . when called upon to explain the transition from finite to infinitesimal magnitudes, he [Leibniz —ed.] resorted to a quasi-philosophical principle known as the law of continuity. We have seen previous applications made of this doctrine by Kepler and by Nicholas of Cusa. The latter may have influenced Leibniz in this respect, as well as in the philosophical doctrine of monads.[16]

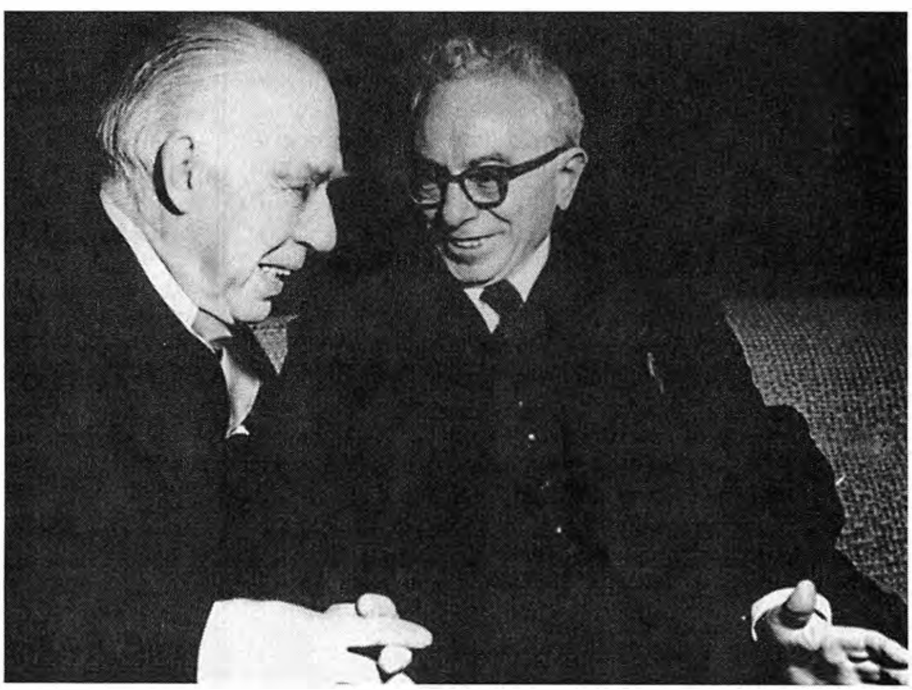
Later Boyer says:

Leibniz justified the limiting condition by the law of continuity, whereas mathematics has since shown that the latter must itself first be defined in terms of limits. In this manner of thinking Leibniz seems still to be striving to make use of a vague idea of continuity which we feel we possess and which had bothered thinkers since the Greek period.[17]

Leibniz vs. Cauchy Empiricism

The above description of the Cauchy method by Boyer introduces the empiricist outlook, an outlook that culminated in the mind-destroying New Math of the 1970s. All hypothesis-formation is eliminated. The student is forced to go through pages and pages of definitions of sets of points and axioms of the number system before he even learns about the derivative. Bertrand Russell, the person philosophically responsible for the New Math, had an intense dislike for Leibniz because of Leibniz's assertion of universals. Does knowledge depend, as Russell said, upon induction from particulars, or, do universals exist? Continuity is a universal. So is substance. The empiricist says: "Can you prove there is something real that you can call continuity? Relative to what?" Leibniz successfully and hubristically introduced the idea of continuity into physics and mathematics. He described it as

a principle of general order which I have observed. . . . This principle has its origin in the infinite, and is absolutely necessary in geometry, but it is effective in physics as well, because the sovereign wisdom, the source of all things, acts as a perfect geometrician, observing a harmony to which nothing can be added. This is why the principle serves one as a test or criterion by which to reveal the error of an ill-conceived opinion at once, and from the outside, even before a penetrating internal examination is begun. When the difference between two instances in a given series, or that which is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought, or in their results, must of necessity also be reduced, or become less than any given quantity whatever. Or, to put it more commonly, when two instances or data approach each other continuously, so that one at least passes over into the other, it is necessary for their consequences or results (or the



Niels Bohr Archive, courtesy AIP Emilio Segrè Visual Archives

***Boring and Boring-er**: Twentieth century physicist Niels Henrik Bohr (l.) and mathematician Richard Courant. Courant's desecration of Leibniz's discovery, quoted within, and Bohr's insistence on irrationality as the foundation of quantum physics have been two of the greatest contributions to the destruction of the Western tradition of scientific discovery in this century.*

unknowns) to do so also. This depends on a more general principle: that, as the data are ordered, so the unknowns are ordered also.

In the case of the tangent, the slopes, which are the unknowns, must yield the value at the point in question, i.e. the tangent at the point, when the data, that is, x and y become sufficiently close to the values of x and y at that point.

Leibniz directly discussed the nature of universals, such as continuity, in 1670, about two years before he began work on the calculus. He had been asked to write an introduction to a book by Marius Nizolius, written in 1553, called *On the True Principles of Philosophy, against Pseudo-Philosophers.* Nizolius, a nominalist, denied that a "universal is anything more than all particulars taken simultaneously and collectively, in Leibniz's words But, Leibniz pointed out, "if universals were nothing but collections of individuals, it would follow that we could attain no knowledge by demonstration—a conclusion which Nizolius actually draws—but only through collecting individuals, or by induction."

The nominalist says: "Induction from experience teaches us that if we put our fingers in the fire they will be burnt." But, without realizing it, Leibniz says, the nominalist is using

> the following universal propositions, which do not depend on induction but on a universal idea or definition of terms: 1. If the cause is the same or similar in all cases, the effect will be the same or similar in all; 2. the existence of a thing which is not senses is not assumed; and finally, 3. whatever is not assumed, is to be disregarded in practice until it is proved.

Thus, continuity is not merely something we infer on the basis of the observed proximity of a set of points. It works the other way. Because the universe obeys the principle of continuity, and because our mind, as part of the universe, obeys this principle, we can make inferences about the way successive points relate to one another, and about the way physical processes must work.

Leibniz made a useful reference to series in this essay when he said:

> Induction in itself produces nothing, not even any moral certainty, without the help of propositions depending, not on induction, but on universal reason. For if these helping propositions too were derived from induction, they would need new helping propositions, and so on to infinity, and moral certainty would never be obtained. By induction alone, we should never perfectly know the proposition that the whole is greater than its part, for someone would soon appear, and for some reason, deny that it is true in cases not yet observed.[18]

Thus, to explain the formation of series of numbers, Leibniz sought the process that generated the entire series. Beginning with the universal principle of identity, he was able to show how one series can be derived from another series. Also, with curves, he saw that there is a single process which generates the whole curve, but which is revealed at each very small interval of the curve. That is the true story of the invention of the calculus.

Ernest Schapiro, M.D., an organizer for Lyndon LaRouche's political movement, was a member of the biological holocaust task force, set up by LaRouche in 1974, and he co-authored two Executive Intelligence Review *Special Reports on the AIDS crisis, produced in the mid-1980s.*

Notes

1. Carl B. Boyer, 1959. *The History of the Calculus and Its Historical Development* (New York: Dover Publications).
2. J.M. Child (translator), 1920. *The Early Mathematical Manuscripts of Leibniz* (Chicago: Open Court Publishing Co.).
3. For the "Art of Combinations," see: Leroy Loemker (editor and translator), *Gottfried Leibniz: Philosophical Papers and Letters* (Chicago: Chicago University Press, 1976), p. 73ff.
4. Child, p. 30
5. Loemker, p. 265
6. For letter to the Grand Duke, see Loemker, p. 261
7. By diagonals are meant the slanted rows 1 1, 1 2 1, 1 3 3 1, 1 4 6 4 1. If you rotate the figure clockwise 45°, you see Pascal's triangle.
8. The reader can construct a three-dimensional model of this as a useful exercise.
9. Loemker, p. 73
10. Child, p. 147
11. Child, p. 150
12. Loemker, p. 447
13. Richard Courant, 1969 *What Is Mathematics?* (New York: Oxford University Press), pp. 398–99.
14. Courant, p. 434
15. Boyer, pp. 6-7
16. Boyer, p. 217
17. Boyer, p. 218
18. Loemker, p. 129